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Determinantal varieties associated to rank two vector bundles on projective spaces and splitting theorems

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0. Introduction

In this paper, we work over an algebraically closed field k. Let E be a rank 2 vector bundle on n-dimensional projective spaces $\mathbf{P}^n_k$.

In [3], H. Sumihiro showed the following theorem in the case char $k = 0$.

Theorem 0.1. *Let P be a 4- or 5-dimensional projective linear subspace of $\mathbf{P}^n_k$ and $\bar{E} = E|P$ be the restriction of E to P. Then $\bar{E}$ splits into line bundles if and only if $H^1(P, \mathcal{E}nd(\bar{E})) = 0$.*

The aim of this article is to prove that this theorem holds affirmatively true in char $k = p > 0$. The proof is almost same with the one of char $k = 0$, namely, is obtained by studying some geometric structures of the Hilbert scheme of $\mathbf{P}^n_k$ at determinantal subvarieties. In char $k = p > 0$, however, since we can not use Kodaira vanishing theorem and Le-Potier vanishing theorem, we have to observe some vanishings of cohomologies appeared in [3] carefully.

1. Determinantal Varieties

We first recall the definition and some properties of determinantal varieties associated to 2-bundles (cf. [3]).

1.1. **Definition of determinantal varieties.** Let E be a rank 2 vector bundle on an $\mathbf{P}^n_k$, $\pi : P(E) \to \mathbf{P}^n_k$ the projective bundle associated to E over $\mathbf{P}^n_k$, L_E the tautological line bundle on $P(E)$ and let $G = \mathrm{Grass}(H^0(E), m+1)$ be the Grassmann variety which parametrizes $(m+1)$-dimensional linear subspaces of $H^0(\mathbf{P}^n_k, E)$. We assume that E is very ample, i.e., L_E is a very ample line bundle. Then we can take $s = \langle s_1, s_2, \dots, s_{m+1}\rangle \in G$ ($s_i \in H^0(\mathbf{P}^n_k, E)$) with $n = 2m$ (resp. $n = 2m+1$) satisfying the following condition

$(*)$
1) $Y = Y_s = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a smooth closed subscheme of $P(E)$ of pure codimension $m+1$,
2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \emptyset$,

where D_i is the tautological divisor on $P(E)$ defined by s_i and $W(s_i)$ is the zero locus on $\mathbf{P}^n_k$ of s_i ($1 \le i \le m+1$).

Let $X_s = \pi(Y_s)$. Then we can show that X_s is a closed subscheme of $\mathbf{P}^n_k$ which is isomorphic to Y_s through π with the following defining equations:

$$s_i \wedge s_j = 0 \quad (1 \le i \le j \le m+1).$$

Definition 1.1. We call the closed subscheme $X = X_s$ of $\mathbf{P}^n_k$ the (smooth) determinantal variety associated to E defined by $s \in G$.

Remark 1.1. X depends on the choice of $s \in G$ subject to the condition $(*)$.

As for determinantal varieties, we obtain the following.

Theorem 1.1. *Let the notaion be as above.*

1) *$U = \{s \in G | s$ satisfies the condition $(*)\}$ is a Zariski open subset of G.*
2) *There exists a closed subscheme Ξ of $\mathbf{P}_k^n \times U$ such that the second projection $q : \Xi \subset \mathbf{P}_k^n \times U \to U$ is faithfully flat and $X_s = q^{-1}(s)$ for any $s \in U$. Thus smooth determinantal varieties associated to E form a smooth family over an open subset of G and hence they are diffeomorphic to each other.*

When $n = 4$ or 5, let I_X be the defining ideal of a determinantal subvariety X in $\mathbf{P}^n$. Then I_X has the following resolution of vector bundles.

Lemma 1.2. *In above notaion, there exists on an exact sequence*

$$0 \longrightarrow E^*(-c_1) \longrightarrow \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}(-c_1) \longrightarrow I_X \longrightarrow 0,$$

where c_1 is the first Chern number of E.

1.2. **Tangent bundle and normal bundle of determinantal varieties.** In this subsection, we consider when $n = 4$ or 5, i.e., $m = 2$.

1.2.1. Let E be a very ample rank two bundle on $\mathbf{P}_k^n$ and X a determinantal variety associated to E. Let H be the restriction of a hyperplane of $\mathbf{P}^n$ to X and D the restriction of a tautological divisor of $P(E)$ to X through the isomorphism π.

Then we can obtain the following diagram of exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & T_{P(E)/\mathbf{P}^n}|Y \arrow[r, "\sim"] \arrow[d] & \mathcal{O}_X(2D - c_1H) \arrow[d, "\alpha"] & \\
0 \arrow[r] & T_Y \arrow[r] \arrow[d, equal] & T_{P(E)}|Y \arrow[r] \arrow[d] & N_{Y/P(E)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_X \arrow[r] & T_{\mathbf{P}^n}|X \arrow[r] \arrow[d] & N_{X/\mathbf{P}^n} \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 &
\end{tikzcd}$$

where α is an injection induced by the snake lemma. Since $N_{Y/P(E)} \simeq \bigoplus^3 \mathcal{O}_X(D)$ by the condition of Y, we obtain the following.

Proposition 1.3. *There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(2D - c_1H) \longrightarrow \bigoplus^3 \mathcal{O}_X(D) \longrightarrow N_{X/\mathbf{P}^n} \longrightarrow 0.$$

1.2.2. From the exact sequence of the above proposition, we have the following exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_X(2D - c_1H)) \to \bigoplus^3 H^0(\mathcal{O}_X(D)) \to H^0(N_{X/\mathbf{P}^n}) \\ &\to H^1(\mathcal{O}_X(2D - c_1H)) \to \bigoplus^3 H^1(\mathcal{O}_X(D)). \end{aligned}$$

Now we recall $Y = D_1 \cap D_2 \cap D_3$. Consider the canonical exact sequence

$$(*)_1 \qquad 0 \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{P(E)}(2D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to 0,$$

from which we obtain the following exact sequence:

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{P(E)}(D - c_1H)) \to H^0(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^0(\mathcal{O}_{D_1}(2D - c_1H)) \\ &\to H^1(\mathcal{O}_{P(E)}(D - c_1H)) \to H^1(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^1(\mathcal{O}_{D_1}(2D - c_1H)) \\ &\to H^2(\mathcal{O}_{P(E)}(D - c_1H)). \end{aligned}$$

Since $H^i(D - c_1H) = H^i(E^*)$ $(0 \leq i \leq 4)$ and we can show that $H^0(E^*) = 0$ and $H^1(E^*) = H^{n-1}(E \otimes K_{\mathbf{P}^n}) = 0$, where $K_{\mathbf{P}^n}$ is the canonical divisor of $\mathbf{P}^n$, it turns out that $H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1}(2D - c_1H))$ $(i = 0, 1)$ if $H^2(E^*) = H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$.

Considering the exact sequences similarly

$$(*)_2 \qquad \begin{aligned} &0 \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to 0, \end{aligned}$$

$$(*)_3 \qquad \begin{aligned} &0 \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to \mathcal{O}_Y(2D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{D_1}(-c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{P(E)}(-D - c_1H) \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{D_1}(-c_1H) \to 0, \end{aligned}$$

we obtain isomorphisms $H^i(\mathcal{O}_{D_1}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H))$ and $H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H)) \simeq H^i(\mathcal{O}_Y(2D - c_1H))$ $(i = 0, 1)$. Summing up the above, we conclude that $H^i(\mathcal{O}_X(2D - c_1H)) \simeq H^i(\mathbf{P}^n, S^2(E)(-c_1))$.

On the other hand, since there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{E}nd(E) \longrightarrow S^2(E)(-c_1) \longrightarrow 0,$$

we have a canonical isomorphism $H^1(S^2(E)(-c_1)) \simeq H^1(\mathcal{E}nd(E))$ and $\dim H^0(S^2(E)(-c_1)) = \dim H^0(\mathcal{E}nd(E)) - 1$.

In addition we easily see that $\dim H^0(\mathcal{O}_X(D)) = \dim H^0(E) - 3$.

Summarizing the above, we get the following proposition.

Proposition 1.4. *Assume that $n = 4$ or 5 and $H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$. If $H^1(\mathcal{E}nd(E)) = 0$, then*

$$\dim H^0(N_{X/\mathbf{P}^n}) = 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1$$

Remark 1.2. When char $k = 0$, we get $H^i(E^*) \simeq H^{n-i}(E \otimes K_{\mathbf{P}^n_k}) = 0$ for $0 \leq i \leq n - 2$ by Le-Potier vanishing theorem. So we do not need the assumption $H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$ in the above proposition. Also the proof itself becomes slightly simpler because we can use the vanishing theorems.

2. Hilbert Schemes

In this section, we assume that $n = 4$ or 5. Let $\mathcal{H}ilb$ be the Hilbert scheme of $\mathbf{P}^n$.

2.1. Let $\varphi : U \ni s \mapsto X_s \in \mathcal{H}ilb$ be the morphism induced by Theorem 1.1. Let $\mathrm{Aut}(E)$ be the automorphism group of E. Then $\mathrm{Aut}(E)$ is a reduced connected linear algebraic group of dimension $\dim H^0(\mathcal{E}nd(E))$.

For every element $g \in \mathrm{Aut}(E)$ and $s = \langle s_1, s_2, s_3 \rangle \in G$, we define

$$g \cdot s = \langle g(s_1), g(s_2), g(s_3) \rangle,$$

where $g(s_i)$ is the composite of s_i with g. Then it defines an action of $\mathrm{Aut}(E)$ on G and we have

$$g \cdot s_i \wedge g \cdot s_j = \det g \; s_i \wedge s_j (1 \leq i \leq j \leq 3),$$

where $\det : \mathrm{Aut}(E) \ni g \mapsto \det(g) \in k^* = k \setminus \{0\}$ is the determinant character. Hence $X_{g \cdot s} = X_s$. Therefore $\mathrm{Aut}(E)$ acts on U and φ is an orbit morphism, i.e., φ is constant on any orbit $O(s) = \{g \cdot s | g \in \mathrm{Aut}(E)\}$.

Then we have the following.

Lemma 2.1. *The stabilizer* $\mathrm{Stab}(s)$ *of* $s \in U$ *is equal to the multiplicative group* k^*.

As a trivial corollary of the above lemma and Proposition 1.3, we observe the following.

a) Every orbit has the same dimension $\dim \mathrm{Aut}(E)/k^*$. Hence the action of $\mathrm{Aut}(E)$ on U is closed, i.e., every orbit is closed in U.
b) $\dim O(s) = \dim H^0(\mathcal{E}nd(E)) - 1$

Proposition 2.2. *Under the same assumptions in Proposition 1.4, if* $H^1(\mathcal{E}nd(E)) = 0$ *then*

$$\dim \overline{\varphi(U)} = \dim H^0(N_{X_s/\mathbf{P}^n}).$$

Proof. Using the exact sequence in Proposition 1.3, we see that $\varphi^{-1}(\varphi(s))$ $(s \in U)$ consists of finitely many orbits. Hence

$$\begin{aligned}
\dim \overline{\varphi(U)} &= \dim U - \dim O(s) \\
&= \dim \mathrm{Grass}(H^0(E), 3) - \dim H^0(\mathcal{E}nd(E)) + 1 \\
&= 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1.
\end{aligned}$$

So the result follows by Proposition 1.4. □

2.2. Let $\mathcal{H}ilb^0$ be an irreducible component of $\mathcal{H}ilb$ containing $\overline{\varphi(U)}$ and $T_{X_s, \mathcal{H}ilb}$ the Zariski tangent space of $\mathcal{H}ilb$ at X_s. Then it is known that $T_{X_s,\mathcal{H}ilb} \simeq H^0(N_{X_s/\mathbf{P}^n})$. So we have the following proposition from Proposition 2.2.

Proposition 2.3. *Under the same assumptions in Proposition 1.4, if* $H^1(\mathcal{E}nd(E)) = 0$ *then*

1) $\mathcal{H}ilb^0$ *coincides with* $\overline{\varphi(U)}$.
2) $\mathcal{H}ilb^0$ *is smooth at the determinantal subvarieties associated to* E.

3. Proof of Theorem

Let $\mathrm{PGL}(n+1,k)$ be the automorphism group of $\mathbf{P}^n$ and let $T_\sigma : \mathbf{P}^n \ni x \mapsto \sigma x \in \mathbf{P}^n$ be the transformation of $\mathbf{P}^n$ defined by $\sigma \in \mathrm{PGL}(n+1,k)$.

Since it is well-known that E splits if and only if $\bar{E}$ splits, we may assume that E is a rank two vector bundle on $\mathbf{P}^n$ (n being either 4 or 5). In addition after multiplying E by a suitable line bundle, we may assume that E is a very ample vector bundle enjoying the assumption in Proposition 1.4.

Suppose that $H^1(\mathcal{E}nd(E)) = 0$. Hence it follows from Proposition 2.3 that $\sigma\overline{\varphi(U)} = \overline{\varphi(U)}$ for every element $\sigma \in \mathrm{PGL}(n+1,k)$. Since $\varphi(U)$ is a constructible set, there exist two element s, $t \in U$ satisfying $X_{\sigma^*(s)} = X_t$, where $X_{\sigma^*(s)}$ is the determinantal subvariey associated to $T^*_\sigma(E)$ defined by $\sigma^*(s) = \langle T^*_\sigma(s_1), T^*_\sigma(s_2), T^*_\sigma(s_3)\rangle$. Consider the resolutions of the defining ideal sheaves I_{X_t} of X_t and $I_{X_{\sigma^*(s)}}$ of $X_{\sigma^*(s)}$ respectively (cf. Lemma 1.2):

$$(**)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & E^* & \longrightarrow & \overset{3}{\bigoplus}\mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_t}\otimes\mathcal{O}(c_1) & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & T^*_\sigma(E^*) & \longrightarrow & \overset{3}{\bigoplus}\mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_{\sigma^*(s)}}\otimes\mathcal{O}(c_1) & \longrightarrow & 0. \end{array}$$

Then it is observed that there exists an isomorphism $\psi : \overset{3}{\bigoplus}\mathcal{O}_{\mathbf{P}^n} \to \overset{3}{\bigoplus}\mathcal{O}_{\mathbf{P}^n}$ such that ψ makes the diagram in $(**)$ commutative and so we see that $T^*_\sigma(E)$ is isomorphic to E, i.e., E is a homogeneous vector bundle. Since every homogeneous bundle on $\mathbf{P}^n$ of rank $r < n$ is a direct sum of line bundles even if char $k = p > 0$ (cf. [2]), we can complete the proof of Theorem 0.1.

References

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